Lecture notes
"Stochastic modelling and derivatives
in traditional markets and in crypto-markets "
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Lecture 1

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Notations

We repeatedly use the same notations.

- Interest rate: $(r_u : u \geq 0)$
- Discount factor $D(t,T) = e^{-\int_t^T r_u \mathrm{d}u}$
- (riskless = non defaultable) Zero-coupon bond at time t for the maturity T: $B(t,T)$
- S for the cash-price
- $\mathtt{C}_t(\phi_T, T)$ for the cash-price (prix au comptant) at time t for the cashflow Φ_T delivered at T
- $\mathtt{F}_t(\phi_T, T)$ for the forward price (prix forward) at time t (and paid at T) for the cashflow Φ_T delivered at T
- $\mathtt{Fut}_t(\phi_T, T)$ for the Future price on Φ_T delivered at T

I Introduction to Pricing and Hedging

I-a Valuation without model

In the financial markets, vanilla options are contracts that give to their owners the right but not the obligation to sell or buy the underlying stock in the future. Vanilla options are usually traded on organized markets and thus, their prices are public and obey to the supply-demand rule (bid-offer); their observations reveal anticipations from market agents and can be used to infer on the risk models. More complex derivatives (called exotic) are Over-The-Counter products and their prices are usually not publicly available on trading plateforms. Knowing how to fix the prices of these derivatives is of great importance.

All these prices obey to many rules and among them the important rule of **no-arbitrage**. In the market, we can find a special kind of traders whose role is to detect the opportunities of making a riskless profit as soon as it arises. Once the arbitrages are detected, the traders simultaneously enter into transactions in two or more markets making advantage of the differences in prices. Hence, the intervention of the arbitrageurs eliminate the arbitrages, the **no-arbitrage** is therefore a realistic modeling assumption.

I-a.1 Absence of arbitrage opportunity

The notion of arbitrage which is at the origin of the theory of pricing is fundamental to understand. In particular, it is important to verify that when the theory is applied, the conditions of **no-arbitrage** are <u>almost verified</u> in the market in which we are interested.

▷ **Definition of a frictionless market.** A **frictionless** market is a theoretical trading environment where the costs and limits associated with transactions do not exist. The **frictionless** market assumptions are important to consider since actual costs will be associated with real world applications.

Definition I.1 (Frictionless market)**.** *We consider that a market is* ***frictionless*** *if the following assumptions are verified.*

- *There are no transaction, no trade costs, no exchange fees nor liquidity costs.*
- *There are no taxes on transactions and gains.*
- *There is no difference between the ask and the bid price, which means that the spread is zero.*
- *The transactions are instantaneous.*
- *The negotiated assets are very liquid and perfectly divisible.*
- *There are no limit on short sales.*

- *The market participants are price takers and small investors have no impact on the prices. The price is also assumed to be exogenous.*
- *The interest rates for lending or borrowing are the same.*

▷ **Definition of no-arbitrage.** We will introduce the notion of **no-arbitrage**.

Definition I.2 (No-arbitrage)**.** *An arbitrage is a self-financing strategy that is worth zero initially and yields a positive gain without any risk (non-negative almost surely and (strictly)-positive with positive probability).*

On the opposite, the absence of arbitrage opportunity or ***no-arbitrage*** *means that a (almost-sure) profit can not be generated from a zero initial investment.*

We will refer the consequence of this assumption as $\stackrel{\text{N.A.}}{\Longrightarrow}$.

▷ **Definition of unique-price rule.**

Corollary I.3 (Unique-price principle)**.** *Under the assumption of* ***no-arbitrage****, if the values of two portfolios coincide at a given date, then these two portfolios have the same value at any other intermediate date.*

In a variant of this corollary, we derive $(\forall\omega\in\Omega)V_T^1 \le V_T^2$, then $V_t^1 \le V_t^2 \;\; \forall t \le T$.

Proof. We consider two portfolios of respective values V_1 and V_2. Suppose that $V_t^1(\omega) > V_t^2(\omega)$ in some scenarios $\omega \in A$, and that $V_T^1(\omega) = V_T^2(\omega)$ for all $\omega \in \Omega$. We consider a global Portfolio V and show that an arbitrage can be made using this portfolio, when events in A occur. We consider that short sales as allowed.

- If $\omega \in A$, we construct V by :
  - short selling portfolio (1) at the price V_t^1.
  - buying portfolio (2) at the price V_t^2.
  - investing the cash $V_t^1 - V_t^2$ in the riskless asset, which means that at time T we will have 1 capitalized € multiplied by $V_t^1 - V_t^2$, we will denote the capitalized euro by 1^c. We assume that the institution holding the liquidity does not run any default risk, which means that we will certainly obtain the cash-flow $1^c \times (V_t^1 - V_t^2) > 0$ at time T. At $t = 0$, the value of the portfolio is $V_t = (V_t^1 - V_t^2) - V_t^1 + V_t^2 = 0$ € .
- If $\omega \notin A$, we do not need to make any investment $V_t = 0$ € .

Then we adopt a **static strategy** until we reach T, which means that our position on the assets remains the same.

- If $\omega \notin A$, we have $V_T = 0$ € .

Strategy	Portfolio value at t in €	Portfolio value at T in €
$\omega \in A$	$V_t = (V_t^1 - V_t^2)[\text{cash}]$ $-V_t^1[\text{asset 1}]$ $+V_t^2[\text{asset 2}] = 0$	$V_T = (V_t^1 - V_t^2) \times 1^c > 0$
$\omega \notin A$	$V_t = 0$	$V_T = 0$

Table 1: Arbitrage opportunity made on A, using V

- If $\omega \in A$, we clear out the global position $V_T = -V_T^1 + V_T^2 + (V_t^1 - V_t^2) \times 1^c$ € $= (V_t^1 - V_t^2) \times 1^c$ € > 0 .

An arbitrage is made on A.

□

I-a.2 Impact of no-arbitrage on the prices of derivatives

In the financial market, the prices of many financial products can be deduced without a reference to any model.

▷ **Price of Forward contracts.** We assume that the prices of our instruments are expressed in € . We recall that $\mathtt{F}_t(S_T, T)$ stands for the forward price at time t (and paid at T) for the cash-flow S_T delivered at T. This price is different from the cash-price $\mathtt{C}_t(S_T, T)$ delivered at T and for which the payment is immediate. We would like to find a relation between these two prices and the price of a (non-defaultable) Zero-coupon Bond of maturity T, which we will denote by $B(t, T)$. This instrument allows to gain 1 € at T.

Proposition I.4 (Forward Price)**.**

$$\mathtt{C}_t(S_T, T) = B(t, T)\mathtt{F}_t(S_T, T).$$

We notice that this relation will be seen to hold regardless of any choice of modeling, which means that it is model-free.

Proof. A static arbitrage argument allows to compare the price of the forward contract to the spot price of the asset S at the date t. In order to guarantee the holding of the asset S at T, we have two possible strategies that we will detail bellow.

- Strategy (1): Paying at t, the cash-price $\mathtt{C}_t(S_T, T)$ in order to receive S_T at T. $V_t^1 = \mathtt{C}_t(S_T, T)$, $V_T^1 = S_T$.
- Strategy (2): Buying a Forward contract on S between t and T. The price to pay at T is $\mathtt{F}_t(S_T, T)$ € .

In order to be able to pay this price at T, we need to place in the bank an amount of money that will guarantee to have $\mathtt{F}_t(S_T, T)$ at T. The financial instrument adapted to this kind of situations is, by definition, the (non-defaultable) Zero-coupon bond of maturity T, which price is $B(t, T)$. Thus, the price to pay at t is $V_t^2 = \mathtt{F}_t(S_T, T)B(t, T)$ € , and $V_T^2 = S_T$.

Strategy	Portfolio value at t in €	Portfolio value at T in €
(1)	$V_t^1 = \mathtt{C}_t(S_T, T)$	$V_T^1 = S_T$
(2)	$V_t^2 = \mathtt{F}_t(S_T, T)B(t, T)$	$V_T^2 = S_T$

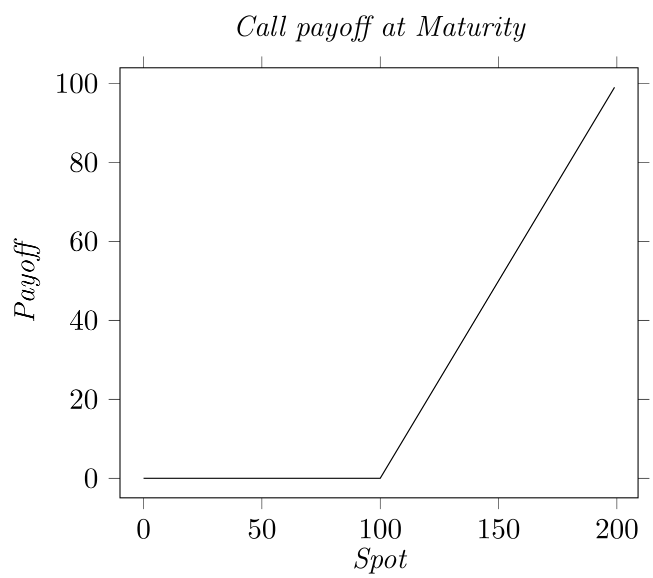
Table 2: Cashflows of the two strategies at times t and T

Since the two portfolios have the same value at T, we conclude from the **no-arbitrage** assumption that $\mathtt{F}_t(S_T, T)B(t, T) = \mathtt{C}_t(S_T, T)$ for all $t \leq T$. □

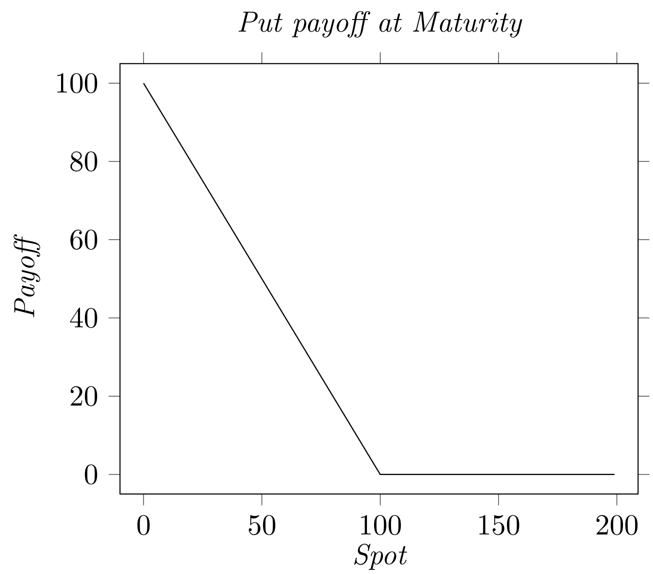
Remark I.5. If S is an asset traded in the market and does not pay dividends between before the maturity T, then $S_t = \mathtt{C}_t(S_T, T)$. In general, if S provides dividends between t and T, then $S_t \geq \mathtt{C}_t(S_T, T)$ and this inequality is strict if the dividends are strictly positive.

▷ **Put-Call Parity.**

Definition I.6 (Call Option). *A Call option on S of strike K and maturity T is a contract giving the right to buy the asset S at price K, at T. This contract allows the protection from the increase of prices of the asset S . The payoff of this option is $(S_T - K)_+ = \max(S_T - K, 0)$.*



Definition I.7 (Put Option)**.** *A put option on S of strike K and maturity T is a contract giving the right to sell the asset S at price K, at T. This contract allows assurance against the fall of prices of the asset S. The payoff of this option is $(K - S_T)_+ = \max(K - S_T, 0)$.*



In the following sections, we denote by $\texttt{Call}_t(T, K)$ and $\texttt{Put}_t(T, K)$ the respective cash-prices at time t of Call and Put contracts on S, with characteristics (T, K):

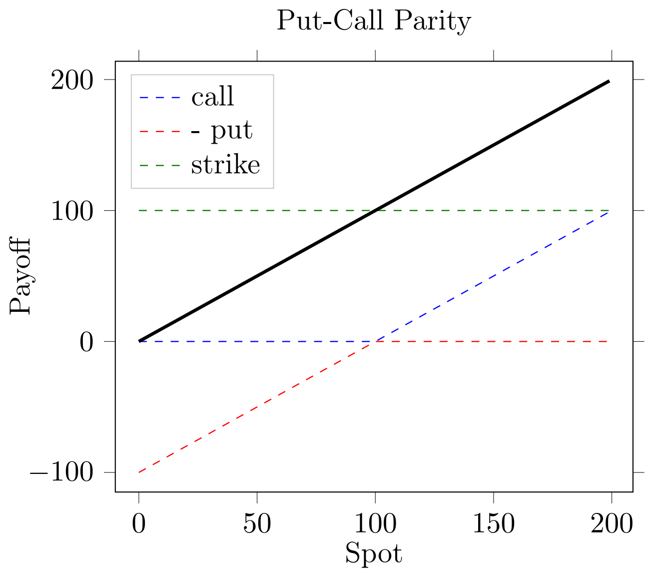
$$\texttt{Call}_t(T, K) := \texttt{C}_t\big((S_T - K)_+, T\big) \qquad \text{and} \qquad \texttt{Put}_t(T, K) := \texttt{C}_t\big((K - S_T)_+, T\big).$$

Proposition I.8 (Put-Call Parity)**.**

$$\texttt{Call}_t(T, K) - \texttt{Put}_t(T, K) = \big(\texttt{F}_t(S_T, T) - K\big)B(t, T)$$

When the asset does not pay dividends between t and T, this expression becomes $\texttt{Call}_t(T, K) - \texttt{Put}_t(T, K) = S_t - KB(t, T)$.

This relation can be interpreted as follows: loosely speaking, the holding of a Call and the sale of a put with the same characteristics guarantee to have at time T the value of the asset S and to sell of the exercise price K. This portfolio can also be obtained by buying the share, at t and reimbursing $KB(t, T)$ at t.



Proof. We recall that the **unique-price principle** holds similarly for cash-prices and forward prices.
In order to prove the Put-Call parity, we can use two different investment strategies.

- Strategy (1): At the date t, we buy the forward contract of maturity T on $(S_T - K)_+$ and we sell the forward contract of maturity T on $(K - S_T)_+$.
- Strategy (2): At the date t, we buy the forward contract of maturity T on the asset S, and we sell the forward contract of maturity T on the strike K.

At the date T, the cash-flows of the two strategies are equal and their value is:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

Strategy	Portfolio value at t in €	Portfolio value at T in €
(1)	$V_t^1 = \mathtt{F}_t\big((S_T - K)_+, T\big) - \mathtt{F}_t\big((K - S_T)_+, T\big)$	$V_T^1 = S_T - K$
(2)	$V_t^2 = \mathtt{F}_t(S_T, T) - \mathtt{F}_t(K, T)$	$V_T^2 = S_T - K$

Table 3: Cashflows of the two strategies at times t and T

By **no-arbitrage**, the values of the two forward prices of the two portfolios are equal at time t.
We know that the forward prices of $\mathtt{Call}(T, K)$ and $\mathtt{Put}(T, K)$ can be expressed using their cash-prices

$$\mathtt{F}_t\big((S_T - K)_+, T\big) = \frac{\mathtt{Call}_t(T, K)}{B(t, T)}, \qquad \mathtt{F}_t\big((K - S_T)_+, T\big) = \frac{\mathtt{Put}_t(T, K)}{B(t, T)}.$$

Thus,

$$\mathtt{F}_t\big((S_T-K)_+,T\big)-\mathtt{F}_t\big((K-S_T)_+,T\big)=\mathtt{F}_t(S_T,T)-\mathtt{F}_t(K,T)=\mathtt{F}_t(S_T,T)-K.$$

Finally, $\mathtt{Call}_t(T,K)-\mathtt{Put}_t(T,K)=\big(\mathtt{F}_t(S_T,T)-\mathtt{F}_t(K,T)\big)B(t,T)=\big(\mathtt{F}_t(S_T,T)-K\big)B(t,T)$. $\square$

If the underlying asset does not pay out some dividends, then $\mathtt{F}_t(S_T,T)=S_t$ and the call-put parity relation is quite explicit.

In the case of dividends, it is not the case; one possibility to get this value is to regress $\mathtt{Call}_t(T,K)-\mathtt{Put}_t(T,K)$ (obtained from market data) on K, in order to extract the forward price $\mathtt{F}_t(S_T,T)$ (implied by the market).

Alternatively, under some dividend model assumptions, $\mathtt{F}_t(S_T,T)$ can be computed (and then calibrated to the market).

$\triangleright$ **Size Effect.** One of the most important consequences of the assumption of the frictionless market is the linearity of the pricing. However, this breaks down if one accounts the price impact by the big investors. For instance, if we want to obtain $10^6 S_T$ at T, it does not necessarily imply that we need to purchase 10^6 Forward contracts at price $\mathtt{F}_t(S_T,T)$, it can be much more costly. In fact, the price of a Forward contract should vary with the size of the transaction. Thus, the big investors have a big impact on the price of the contract.

$\triangleright$ **Model-free bounds of Calls and Puts.** As with forward contracts, the absence of arbitrage makes it possible to establish constraints on the prices of derivatives according to their future characteristics. We will be particularly interested in the properties of the Call and Put options that we have described above, but it will be convenient for the rest to state more general properties regarding the price.

Proposition I.9 (Bounds/properties on call)**.** *We have the following bounds or properties on Call options prices:*

1. *Hedge relation :* $\big(\mathtt{F}_t(S_T,T)-K\big)_+B(t,T)\le\mathtt{Call}_t(T,K)\le\mathtt{F}_t(S_T,T)B(t,T)=\mathtt{C}_t(S_T,T)$.
2. *Bull Spread Relation :* $K\mapsto\mathtt{Call}_t(T,K)$ *is non-increasing.*
3. *Butterfly Spread Relation :* $K\mapsto\mathtt{Call}_t(T,K)$ *is convex.*
4. *Calendar Spread Relation :* $T\mapsto\mathtt{Call}_t(T,K)$ *is non-decreasing, when the underlying security pays no dividends and when* $B(t,T)\le 1$ *(automatic if non-negative interest rate).*

If the first and second derivatives of the Call in strike exist, and that the first derivative in maturity exists as well, the three last properties simply write $\partial_K\mathtt{Call}_t(t,K)\le 0,\partial_K^2\mathtt{Call}_t(T,K)\ge 0,\partial_T\mathtt{Call}_t(T,K)\ge 0$.

Proof. 1. Let us show that (1): $\big(\mathtt{F}_t(S_T,T)-K\big)_+ B(t,T) \le \mathtt{Call}_t(T,K)$.
$(S_T-K)_+ \ge 0$ and $(K-S_T)_+ \ge 0 \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T,K) \ge 0$ and $\mathtt{Put}_t(T,K) \ge 0$.
Hence, using the Put-Call parity: $\mathtt{Call}_t(T,K) = \mathtt{Put}_t(T,K) + \big(\mathtt{F}_t(t,T)-K\big)B(t,T) \ge \big(\mathtt{F}_t(t,T)-K\big)B(t,T)$.

Let us show that (2): $\mathtt{Call}_t(T,K) \le \mathtt{F}_t(S_T,T)B(t,T) = \mathtt{C}_t(S_T,T)$.
$(S_T-K)_+ \le S_T \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(K,T) \le \mathtt{C}_t(S_T,T) = \mathtt{F}_t(S_T,T)B(t,T)$.

2. Let us show that the Call price is non-increasing in strike. Let K_1 and K_2 be two exercice prices such as $K_1 > K_2$. For a given $S_T(\omega)$, we have $(S_T-K_1)_+ \le (S_T-K_2)_+ \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T,K_1) \le \mathtt{Call}_t(T,K_2)$.

3. The payoff of the Call $K \to (S_T-K)_+$ is a convex function of the strike, by definition : $\forall \varepsilon > 0 \ \ \big(S_T-(K-\varepsilon)\big)_+ - 2(S_T-K)_+ + \big(S_T-(K+\varepsilon)\big)_+ \ge 0$. Using the **no-arbitrage** assumption, at time t the cash-prices for the Calls of respective strikes K, $K-\varepsilon$ and $K+\varepsilon$ verify $\mathtt{Call}_t(T,K-\varepsilon) - 2\mathtt{Call}_t(T,K) + \mathtt{Call}_t(T,K+\varepsilon) \ge 0$, i.e. the convexity property.

4. We would like to show that the Call price is non-decreasing in maturity. Let T_1 and T_2 be two maturities such as $T_1 < T_2$: since $\mathtt{F}_{T_1}(S_{T_2},T_2)B(T_1,T_2) = S_{T_1}$ (no dividend), the hedge relation gives $\mathtt{Call}_{T_1}(T_2,K) \ge \big(S_{T_1} - KB(T_1,T_2)\big)_+ \ge (S_{T_1}-K)_+$ since $B(T_1,T_2) \le 1 \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T_2,K) \ge \mathtt{Call}_t(T_1,K)$.

□

I-a.3 Static replication of a payoff

In the market, for each maturity T there are numerous strikes at which call/put option prices are available. Imagine that there is a continuum of such strikes: intuitively it gives a crucial information about every possible future change in the underlying asset, and thus it should define without ambiguity the price of any derivatives written on the asset at time T. This is the meaning of the following Carr formula.
The main interest of the following result is its use in the replication of some payoffs, for example, the log-payoff in Variance payoff using only Call and Put options.

Lemma I.10. *The system of Call and Put options price of maturity T : $\big(\mathtt{Call}_t(T,k), \mathtt{Put}_t(T,K)\big)_{K\ge 0}$ allow to generate the prices of vanilla options of payoffs $h(S_T)$, where h can be any regular function or a difference of convex functions.*

$$\forall\, x_0, x \ge 0 \quad h(x) = h(x_0) + h'(x_0)(x-x_0) + \int_{x_0}^{+\infty} h''(K)(x-K)_+ \mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+ \mathrm{d}K.$$

Proof. We consider two cases : $x \geq x_0$ and $x < x_0$.
If $x \geq x_0$

$$\begin{aligned}\int_{x_0}^{+\infty} h''(K)(x-K)_+\mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+\mathrm{d}K &= \int_{x_0}^{x} h''(K)(x-K)\mathrm{d}K + 0\\ &= \Big[h'(K)(x-K)\Big]_{K=x_0}^{K=x} + \int_{x_0}^{x} h'(K)\mathrm{d}K\\ &= -h'(x_0)(x-x_0) + h(x) - h(x_0)\\ &= h(x) - h(x_0) - h'(x_0)(x-x_0).\end{aligned}$$

If $x < x_0$

$$\begin{aligned}\int_{x_0}^{+\infty} h''(K)(x-K)_+\mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+\mathrm{d}K &= 0 + \int_{x}^{x_0} h''(K)(K-x)\mathrm{d}K\\ &= \Big[h'(K)(K-x)\Big]_{K=x}^{K=x_0} - \int_{x}^{x_0} h'(K)\mathrm{d}K\\ &= h'(x_0)(x_0-x) - (h(x_0) - h(x))\\ &= h(x) - h(x_0) - h'(x_0)(x-x_0).\end{aligned}$$

□

By taking $x = S_T$, we get the following result.

Theorem I.11 (Carr formula)**.** *Under the **no-arbitrage** assumption, the system of Call and Put options price of maturity T : $\big(\mathtt{Call}_t(T,k), \mathtt{Put}_t(T,K)\big)_{K\geq 0}$ allow to generate the prices of vanilla options of payoffs $h(S_T)$, where h can be any regular function or a difference of convex functions. The cash-price at time t of the payoff $h(S_T)$ is*

$$\begin{aligned}\mathtt{C}_t(h(S_T),T) &= h(x_0)B(t,T) + h'(x_0)\big(\mathtt{C}_t(S_T,T) - x_0B(t,T)\big)\\ &\quad + \int_{x_0}^{+\infty} h''(K)\mathtt{Call}_t(T,K)\mathrm{d}K + \int_0^{x_0} h''(K)\mathtt{Put}_t(T,K)\mathrm{d}K.\end{aligned}$$

We choose as a anchor point x_0 the value of the Forward contract on S of maturity T. The price of the payoff S_T is given by $\mathtt{C}_t(S_T,T) = B(t,T)\mathtt{F}_t(S_T,T) = B(t,T)x_0$. The formula above then becomes

$$\begin{aligned}\mathtt{C}_t(h(S_T,T)) &= h\big(\mathtt{F}_t(S_T,T)\big)B(t,T) + \int_{\mathtt{F}_t(S_T,T)}^{+\infty} h''(K)\mathtt{Call}_t(T,K)\mathrm{d}K\\ &\quad + \int_0^{\mathtt{F}_t(S_T,T)} h''(K)\mathtt{Put}_t(T,K)\mathrm{d}K.\end{aligned}$$

This gives a static replication formula for a payoff $h(S_T,T)$, using call/put with the same maturity. This is valid without reference to a specific model (true for Market price data or for synthetic price): this is model-free.

Note that the strikes of call/put are OTM or ATM.

▷ **Practical use.** The static model-free replication of the payoff h, requires having a continuum of Calls and Puts in K. However, Call and Put options are typically available for a number of different strikes and maturities. In general, for a given underlying, the Call and Put options are only traded for a finite number of maturities : $\{T_i, i = 1, \dots, n\}$ arranged in ascending order. For a given maturity, these options are only negotiated for a finite number of strikes $\{K_{i,j}, j = 1, \dots, d_i\}$. Hence, we need to approximate the integrals using finite sums in the Carr formula (trapezoidal rule, rectangle , etc), and when the option with payoff h has maturity T_i, the objective is to find the best approximation given a set of Calls/Puts with the same maturity T_i: $\big(\texttt{Call}_t(T_i, K_{i,j}), \texttt{Put}_t(T_i, K_{i,j})\big)_{i,j\in\{j=1,\dots,d_i\}}$. Any option of intrinsic value h can be expressed as the limit of a linear combination of Calls and Puts OTM/ATM.

However, this formula of static model-free replication does not allow to price an exotic contract nor path dependent.